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PROBLEMS OF ALMOST EVERYWHERE CONVERGENCE RELATED TO HARMONIC ANALYSIS AND NUMBER THEORY

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ABSTRACT

The object of this paper is to discuss certain methods for studying almost everywhere convergence problems. We consider the generalization of the Riesz-Raikov theorem where the dilation number $\theta > 1$ is not necessarily an integer. It is known (see [B2]) that the averages $(1/N)\sum_1^N f(\theta^n x)$ converge a.e. to $\int_0^1 f dx$ whenever θ is algebraic and f a 1-periodic function on $\mathbf{R}$ satisfying $\int_0^1 |f(x)|^2 dx < \infty$. Here the particular case of rational dilation is treated. The reader is referred to [B2] for the general (algebraic) case.

The following definitive relation between a.e. convergence and algebraic numbers is proved. Let $\{\mu_j\}$ be the sequence of measures

$$\mu_j = \mathop{*}_{k=1}^{j} \left(\frac{1}{2}\delta_{-\xi^k} + \frac{1}{2}\delta_{\xi^k}\right), \qquad \xi = \theta^{-1}$$

converging weak* to the natural measure μ on the Cantor set of dissection ratio θ. Then $f * \mu_j \to f * \mu$ a.e. for all $L^\infty(\mathbf{T})$ functions iff θ is algebraic. This fact depends on [B3] and a variant of Rota's theorem [Ro] on a.e. convergence of certain compositions of operators. Further applications of this result in ergodic theory are presented in the last section of the paper. In section 4, a.e. convergence of Riemann sums of periodic L^2-functions is investigated. It is shown that almost surely $R_n f$ has a logarithmic density, where

$$R_n f(x) = \frac{1}{n}\sum_0^{n-1} f\left(x + \frac{j}{n}\right).$$

This result complements the work of R. Salem on the subject.

1. Introduction

In this paper, I intend to report on progress towards rather classical problems on almost sure convergence. A part of it is written in expository style and summa-

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rizes some recent work of the author on the subject (detailed versions have already been published or will appear elsewhere). The rest of the paper contains new results in this area. All of them have a number theoretic flavor and we approach them from a harmonic analysis point of view. The interest of Fourier analysis in proving maximal inequalities is of course well known and this technique was used over and over again, especially in differentiation problems. Applications in ergodic theory, such as studying the behaviour of ergodic averages of the form

$$\frac{1}{N}\sum_{1}^{N} T^{p(n)}f \tag{1.1}$$

where $p(x) \in \mathbf{Z}(x)$ is a polynomial with integer coefficients and T a measure preserving transformation on some measure space, were discussed in [B1] (see also [Th] for another exposition). Most of the work presented here originates from papers written by R. Salem and collaborators (see [Sa], in particular B.2, B.26, B.30, B.32). It deals with problems such as extending the Riesz–Raikov ergodic theorem from integers to arbitrary dilation, the behaviour of Riemann sums and convolutions of Bernoulli distributions. Our interest goes primarily to the function space $L^\infty(\mathbf{T})$ of bounded measurable 1-periodic functions on $\mathbf{R}$, although the results are generally obtained for the larger class $L^2(\mathbf{T})$ of L^2-functions. Results for the full L^2-space are obtained by combining methods from ergodic or martingale theory with almost orthogonality techniques (generally exploited making use of Fourier transform). This permits us to go beyond the Salem results referred to above, based on the sole use of almost orthogonality and requiring conditions of the form

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2\sigma(n) < \infty \tag{1.2}$$

on the Fourier coefficients of the function f. Here σ is some weight function satisfying $\lim_{n\to\infty} \sigma(n) = \infty$ and depends on the problem.

Next, we state and briefly comment on the main theorems of this paper.

In [B2], the following result is proved.

THEOREM 1.3. *Let f be a 1-periodic measurable function on the real line* $\mathbf{R}$ *satisfying* $\int_0^1 |f|^2dx < \infty$. *If $\theta > 1$ is an algebraic number, then*

$$\frac{1}{N}\sum_{1}^{N} f(\theta^n x) \to \int_0^1 f \tag{1.4}$$

at almost every point $x \in \mathbf{R}$.

At present, I don't know whether the restriction θ algebraic is needed here. When θ is an integer, the previous statement holds for f locally L^1 and is the classical Riesz–Raikov theorem (which is a version of the ergodic theorem). The proof of Theorem 1.3 is the easiest for θ rational and will be reproduced here. This case already illustrates the method very well.

Our next concern is Riemann sums $R_n f$ defined by

$$R_n f(x) = \frac{1}{n}\sum_{0}^{n-1} f\left(x + \frac{j}{n}\right). \tag{1.5}$$

A classical theorem of B. Jessen [Je] asserts that if $q > 1$ is an integer, then, for $f \in L^1(\mathbf{T})$,

$$R_{q^r} f \xrightarrow{r\to\infty} \int_0^1 f \quad \text{a.e.} \tag{1.6}$$

It was shown by W. Rudin [Ru] that (1.6) fails to hold, even for L^∞-functions, if the system $\{q^r\}$ is replaced by the set of all positive integers. In particular, there is no maximal inequality relative to $\{R_n f \mid n = 1,2,\dots\}$ for $f \in L^2(\mathbf{T})$. In [Sa] (B2) considers the means

$$\frac{1}{N}\sum_{1}^{N} R_n f \tag{1.7}$$

and the almost everywhere convergence of these averages. Observe that

$$R_n f(x) = \sum_{n\mid k} \hat{f}(k) e^{2\pi i k x} \tag{1.8}$$

and thus

$$\frac{1}{N}\sum_{1}^{N} R_n f(x) = \sum_{k=-\infty}^{\infty} \frac{d(k;N)}{N} \hat{f}(k) e^{2\pi i k x} \tag{1.9}$$

denoting $d(k;N)$ the number of divisors of k which are bounded by N. This issue is left undecided in [Sa] (B2) and again restrictions of the form (1.2) are imposed. We will prove here the following.

THEOREM 1.10. *Let $f \in L^2(\mathbf{T})$. Then, at almost every point x, $R_n f(x)$ has logarithmic density, i.e.*

$$\frac{1}{\log N} \sum_1^N \frac{1}{n} R_n f \to \int_0^1 f \qquad \text{a.e.} \tag{1.11}$$

Deciding whether Theorem 1.10 is also valid for the usual averages (1.7) seems an interesting and difficult problem.

Although in the statement of Theorem 1.3 the assumption of θ to be algebraic is not known to be necessary, such a number theoretic hypothesis may be of relevance in problems of almost everywhere convergence. This fact is illustrated by the following example. Let $\theta > 1$ be a real number and put $\xi = \theta^{-1}$. Consider the convolutions of Bernoulli distributions

$$\mu_j = \mathop{*}_{k=1}^{j} \left(\frac{1}{2}\, \delta_{-\xi^k} + \frac{1}{2}\, \delta_{\xi^k} \right), \tag{1.12}$$

$$\mu = \mathop{*}_{k=1}^{\infty} \left(\frac{1}{2}\, \delta_{-\xi^k} + \frac{1}{2}\, \delta_{\xi^k} \right). \tag{1.13}$$

Here δ_x stands for the Dirac measure at the point x and the measures may be considered on $\mathbf{R}$ or on $\mathbf{T} = \mathbf{R}/\mathbf{Z}$. We choose the second alternative. We have for their Fourier transforms

$$\hat{\mu}_j(n) = \prod_{k=1}^{j} \cos 2\pi \xi^k n \quad \text{and} \quad \hat{\mu}(n) = \prod_{k=1}^{\infty} \cos 2\pi \xi^k n \tag{1.14}$$

where $n \in \mathbf{Z}$. For $\xi < \frac{1}{2}$, μ is supported by a perfect Cantor set of constant dissection ratio, with points $\epsilon_1 \xi + \epsilon_2 \xi^2 + \cdots + \epsilon_k \xi^k + \cdots$ where $\epsilon_k = \pm 1$. Hence, in this case, μ is singular. The problem for which $\xi > \frac{1}{2}$ the measure μ is continuous, is unsolved.

From (1.14), clearly

$$f * \mu_j \xrightarrow{j \to \infty} f * \mu \quad \text{in } L^2 \quad \text{for } f \in L^2(\mathbf{T}). \tag{1.15}$$

As far as pointwise behaviour is concerned, we will show the following amazing fact.

THEOREM 1.16. *The condition for almost sure convergence for the class of L^2 functions is that ξ should be an algebraic number. More precisely,*

(a) *If ξ is not algebraic, there is $f \in L^\infty(\mathbf{T})$ for which $f * \mu_j$ does not converge almost everywhere.*

(b) *Conversely, if ξ is algebraic, there is a maximal inequality*

$$(1.17) \qquad \left\| \sup_j |f * \mu_j| \right\|_{L^2(\mathbf{T})} \leq C \|f\|_{L^2(\mathbf{T})}$$

and hence $f * \mu$ is the limit of $f * \mu_j$ almost everywhere.

The main tool in proving part (a) is the entropy criterion of [B3]. To get part (b), we will rely on a "new" maximal inequality for certain Fourier multipliers, which relates to Rota's theorem [Ro].

Besides the obvious questions related to Theorems 1.3, 1.10 in previous discussion, it seems to me a natural problem to seek for rather general conditions on a sequence of multipliers $\{\lambda^{(j)} \mid j = 1,2,\dots\}$, $\lambda^{(j)} = (\lambda_n^{(j)})_{n \in \mathbf{Z}}$, for which an L^2-maximal inequality holds, i.e.

$$(1.18) \qquad \left\| \sup_j \left| \sum \lambda_n^{(j)} \hat{f}(n) e^{2\pi i n t} \right| \right\|_2 \leq C \|f\|_2.$$

I restrict myself here mainly to the case where each $\lambda^{(j)}$ is given by the Fourier transform of some probability measure on $\mathbf{T}$. This problem is most likely still too general.

The present paper is in some sense a follow-up of [B4], which was purely expository.

Our reference list is by no means exhaustive for the various contributions on the problems discussed in the paper.

As usual, C will stand for various constants (not necessarily absolute but at least independent of the function).

2. Generalities

Given a sequence of Fourier multipliers $\lambda^{(j)} = (\lambda_n^{(j)})_{n \in \mathbf{Z}}$, the simplest technique to estimate the maximal function

$$(2.1) \qquad Mf = \sup_j \left| \sum_{n=-\infty}^{\infty} \lambda_n^{(j)} \hat{f}(n) e^{2\pi i n t} \right|$$

consists of comparing with another sequence $\mu^{(j)}$, writing

$$(2.2) \qquad Mf \leq M_1 f + \left(\sum_j \left| \sum_n [\lambda_n^{(j)} - \mu_n^{(j)}] \hat{f}(n) e^{2\pi i n t} \right|^2 \right)^{1/2},$$

where

$$(2.3) \qquad M_1 f = \sup_j \left| \sum \mu_n^{(j)} \hat{f}(n) e^{2\pi i n t} \right|.$$

Hence

$$\|Mf\|_2 \le \|M_1 f\|_2 + \sup_{n\in\mathbf{Z}} \Big(\sum_j |\lambda_n^{(j)} - \mu_n^{(j)}|^2\Big)^{1/2} \cdot \|f\|_2 \tag{2.4}$$

for $f \in L^2(\mathbf{T})$.

It suffices thus to control $M_1 f$ and the expressions $\sum_j |\lambda_m^{(j)} - \mu_n^{(j)}|^2$, $n \in \mathbf{Z}$. The two sources for bounded maximal operators M_1 are the Hopf maximal inequality for positive contractions (involved in Theorem 1.3) and the martingale maximal function (appearing in Theorems 1.10, 1.16). Let us recall these results.

Hopf's theorem states that if T is a positive L^1-L^∞ contraction on a probability space, then for $f \in L^p$ $(p > 1)$

$$\Big\|\sup_N \Big|\frac{1}{N}\sum_1^N T^n f\Big|\Big\|_p \le C\|f\|_p. \tag{2.5}$$

Here C is a constant only depending on p.

Similarly, if $\mathbf{E}_N$ is a sequence of refining expectation operators on a probability space, one has

$$\Big\|\sup_N |\mathbf{E}_N f|\Big\|_p \le C\|f\|_p \qquad (p > 1) \tag{2.6}$$

called the martingale maximal inequality.

G. C. Rota [Ro] uses the martingale result and a dilation technique to establish the following fact:

Let $\{T_n\}$ be a sequence of positive operators which are contractions on both L^1 and L^∞ and mapping the constant 1-function to itself. Then the sequence of operators $T_1 T_2 \cdots T_n T_n^* \cdots T_1^*$ yields a bounded maximal operator on L^p, $p > 1$. If in particular the T_n are given by convolution on $\mathbf{T}$ with a probability measure μ_n, one gets the inequality

$$\Big\|\sup_n \Big|\sum_{k=-\infty}^{\infty} \Big(\prod_{j=1}^n |\hat{\mu}_j(k)|^2\Big) \cdot \hat{f}(k) e^{2\pi i k t}\Big|\Big\|_p \le C\|f\|_p. \tag{2.7}$$

We will use this fact later. Here is a direct deduction of (2.7) from (2.6). Denote Ω the infinite product $\mathbf{T} \times \mathbf{T} \times \cdots$ equipped with product measure $\mu = \mu_1 \otimes \mu_2 \otimes \cdots$. Let further $\mathcal{E}_n$ be the expectation operator with respect to the variables $\theta_{n+1}, \theta_{n+2}, \ldots$. We have

$$|\mathcal{E}_n[e^{-2\pi i k \sum \theta_j}]| = \prod_{j \le n} |\hat{\mu}_j(k)|. \tag{2.8}$$

Thus the left number of (2.7) equals

$$\begin{aligned}(2.9)\qquad &\sum_n \left\langle \sum_{k=-\infty}^{\infty} \left(\prod_{j=1}^{n} |\hat{\mu}_j(k)|^2 \right) \hat{f}(k) e^{2\pi ikt}, g_n \right\rangle \\ &= \sum_n \left\langle \sum_k \mathcal{E}_n [e^{-2\pi ik\sum\theta_j}] \hat{f}(k) e^{2\pi ikt}, \sum_k \mathcal{E}_n [e^{-2\pi ik\sum\theta_j}] \hat{g}_n(k) e^{2\pi ikt} \right\rangle\end{aligned}$$

where $\{g_n\}$ is a sequence of functions on $\mathbf{T}$ satisfying

$$(2.10)\qquad \left\| \sum_{n=1}^{\infty} |g_n| \right\|_q \leq 1, \qquad q = \frac{p}{p-1}.$$

Observe that if θ denotes the Ω-variable $(\theta_1, \theta_2, \ldots)$,

$$(2.11)\qquad F(t,\theta) = f\left(t - \sum \theta_j\right) \quad \text{and} \quad G_n(t,\theta) = g_n\left(t - \sum \theta_j\right),$$

the nth term in (2.9) is then given by

$$(2.12)\qquad \langle (I \otimes \mathcal{E}_n)F, (I \otimes \mathcal{E}_n)G_n \rangle.$$

The pairing $\langle\ ,\ \rangle$ refers here to the (t,θ)-variable in $\mathbf{T} \times \Omega$. Thus (2.9) is bounded by

$$(2.13)\qquad \left\| \max_n |(I \otimes \mathcal{E}_n)F| \right\|_{L^p(\mathbf{T}\times\Omega)} \cdot \left\| \sum_{n=1}^{\infty} (I \otimes \mathcal{E}_n)|G_n| \right\|_{L^q(\mathbf{T}\times\Omega)}$$

which, by (2.6) and its dual version, is bounded by

$$(2.14)\qquad C\|F\|_p \left\| \sum_{n=1}^{\infty} |G_n| \right\|_q = C\|f\|_p \cdot \left\| \sum |g_n| \right\|_q \leq C\|f\|_p$$

using (2.10),(2.11). This yields (2.7).

In the introduction, the problem of finding general methods to decide the L^2-boundedness of Mf given by (2.1) was stated. In [B3] an efficient criterion is terms of an entropy condition was obtained. For $f \in L^2(\mathbf{T})$ and $\delta > 0$, let $N_f(\delta)$ stand for the metrical entropy numbers of the set

$$(2.15)\qquad \left\{ \sum_{n=-\infty}^{\infty} \lambda_n^{(j)} \hat{f}(n) e^{2\pi int} \,\middle|\, j = 1,2,\ldots \right\}$$

considered a subset of $L^2(\mathbf{T})$ (i.e. the minimal numbers of δ-balls needed for covering the set, possibly ∞). The following fact is a consequence of the results of [B3].

PROPOSITION 2.16. (a) *A necessary condition for the L^q-maximal inequality is a uniform bound*

$$\delta\sqrt{\log N_f(\delta)} < C \tag{2.17}$$

for $0 < \delta < 1$ and f ranging in the unit ball of $L^2(\mathbf{T})$.

(b) *Almost sure convergence of $\sum_{n=-\infty}^{\infty} \lambda_n^{(j)} \hat{f}(n) e^{2\pi int}$ for $j \to \infty$ and $f \in L^\infty(\mathbf{T})$ implies uniform bounds*

$$N_f(\delta < C(\delta); \qquad \delta > 0 \tag{2.18}$$

when f ranges in the unit ball of $L^2(\mathbf{T})$.

In the present context of convolution operators, Proposition 2.16 is proved by methods of random Fourier series. It has many applications (cf. [B3]) and is also the main ingredient in proving Theorem 1.16(a).

3. Algebraic dilations

We give here the proof of Theorem 1.3 in the special case of θ being rational, $\theta = a/b > 1$, $(a,b) = 1$. The problem of not preserving periodicity when multiplying with θ is already encountered here.

Consider the transformation on 1-periodic functions given by

$$Tf(x) = \sum_{k \in b\mathbf{Z}} \hat{f}(k) e^{2\pi ik\theta x}. \tag{3.1}$$

Obviously, T is a positive contraction on all $L^p(\mathbf{T})$-spaces $(1 \leq p \leq \infty)$ and hence satisfies the maximal inequality (2.5).

Define the following partition of $\mathbf{Z}$:

$$\mathfrak{S}_j = \left\{ n \in \mathbf{Z} \,\middle|\, b = b^k m \text{ where } b \text{ does not divide } m \text{ and } 2^{j-1} \leq k < 2^j \right\} \qquad (j \geq 1), \tag{3.2}$$

$$\mathfrak{S}_0 = \left\{ n \in \mathbf{Z} \,\middle|\, b \text{ does not divide } n \right\}.$$

Write

$$f = \sum_{j \geq 0} f_j \qquad \text{where } f_j(z) = \sum_{n \in \mathfrak{S}_j} \hat{f}(n) e^{2\pi inx}. \tag{3.3}$$

Put

$$A_N f = \frac{1}{N} \sum_1^N f(\theta^n x)$$

and estimate the maximal function as follows:

$$\sup_j |A_{2^j} f| \le \sup_j \left| A_{2^j}\left(\sum_{j'>j} f_{j'} \right) \right| \tag{3.4}$$

$$+$$

$$+ \left(\sum_{j\ge 0} |A_{2^j}(f_j)|^2 \right)^{1/2}$$

$$+ \left(\sum_{j\ge 1} |A_{2^j}(f_{j-1})|^2 \right)^{1/2}$$

$$+$$

$$\vdots$$

$$+$$

$$\left(\sum_{j\ge s} |A_{2^j}(f_{j-s}|^2 \right)^{1/2} \tag{3.5}$$

$$+$$

$$\vdots$$

The proof of Theorem 1.3 reduces to showing an inequality for 1-periodic functions f

$$\int_0^1 \sup_N |A_N f|^2 dx \le C\|f\|_2^2 \tag{3.6}$$

which is a positive problem. Hence, the function f may be taken positive and it suffices to restrict N to the set $\{2^j\}$ in the supremum. Clearly

$$T^n f_{j'}(x) = f_{j'}(\theta^n x) \qquad \text{if } n \le 2^{j'-1} \tag{3.7}$$

and hence

$$A_{2^j}(f_{j'}) = \left(\frac{1}{2^j} \sum_1^{2^j} T^n \right) f_{j'} \qquad \text{if } j < j'. \tag{3.8}$$

It follows from (3.8) that

$$(3.4) = \sup_j \left| \left(\frac{1}{2^j} \sum_1^{2^j} T^n \right) \left(\sum_{j'>j} f_{j'} \right) \right| \le \sup_N \left(\frac{1}{N} \sum_1^N T^n \right) \left[\sup_j \left| \sum_{j'>j} f_{j'} \right| \right]. \tag{3.9}$$

Here

$$\sum_{j'>j} f_{j'} = R_{b^{2^j}}(f) \tag{3.10}$$

where R_n stands for the Riemann sum introduced earlier.

Hence, by (2.5) and Jessen's inequality

$$\left\| \sup_j \left| A_{2^j}\Big(\sum_{j'>j} f_{j'}\Big) \right| \right\|_{L^2(0,1)} \le C \left\| \sup_j |R_{b^j} f| \right\|_{L^2(0,1)} \le C\|f\|_2. \tag{3.11}$$

It remains to estimate for $s \ge 0$ the contribution of the terms (3.5). One has

$$\left\| \Big(\sum_{j\ge s} |A_{2^j}(f_{j-s})|^2 \Big)^{1/2} \right\|_{L^2(0,1)} = \Big(\sum_{j \ge s} \|A_{2^j}(f_{j-s})\|^2_{L^2(0,1)} \Big)^{1/2}. \tag{3.12}$$

Assume shown the following "almost orthogonality" lemma.

LEMMA 3.13. *Let $K > 1$ and $f \in L^2(\mathbf{T})$ such that no frequency $k \in \operatorname{supp} \hat{f}$ is a multiple of b^n for $n \ge N/K$. Then*

$$\|A_N f\|_{L^2(0,1)} \le CK^{-1/4}\|f\|_2. \tag{3.14}$$

Letting $N = 2^j$, $K = 2^{-s}$, one then gets

$$\|A_{2^j}(f_{j-s})\|_{L^2(0,1)} \le C2^{-s/4}\|f_{j-s}\|_2 \tag{3.15}$$

and hence

$$\|(3.11)\|_{L^2(0,1)} \le C\cdot 2^{-s/4}\|f\|_2. \tag{3.16}$$

This series is obviously summable and takes care of the (3.5) terms.

PROOF OF LEMMA 3.13. Fix a sufficiently large constant M (depending on $\theta = a/b$) and partition $[1,N]$ in consecutive intervals $I_0, I_1, \dots, I_R$ of length $\sim N/\sqrt{K}$. Estimate by triangle inequality

$$\left\| \sum_1^N f(\theta^n x) \right\|_{L^2(0,1)} \le M \frac{N}{\sqrt{K}} \|f\|_2 + \sum_{r=M}^R \left\| \sum_{n \in I_r} f(\theta^n x) \right\|_{L^2(0,1)}. \tag{3.17}$$

Consider the function Ω on $\mathbf{R}$ given by

$$\Omega(x) = \left| \int_0^{1/100} e^{2\pi i t x} dt \right|^2. \tag{3.18}$$

This function satisfies $\Omega \ge 0$, $\operatorname{supp} \hat{\Omega} \subset [-1/100, 1/100]$ and $\Omega > c$ on $[0,1]$. Write

$$\left\| \sum_{n\in I_r} f(\theta^n x) \right\|^2_{L^2(0,1)} \le C \int \left| \sum_{n\in I_r} f(\theta^n x) \right|^2 \Omega(x)dx \tag{3.19}$$
$$\le C \sum_{\substack{n,n'\in I_r\\ k,k'\in \mathbf{Z}}} |\hat{f}(k)\|\hat{f}(k')|\hat{\Omega}(k\theta^n - k'\theta^{n'}).$$

For $n \le n'$ in I_r, $r \ge M$, $\hat{\Omega}(k\theta^n - k'\theta^{n'}) \neq 0$ clearly implies

$$|k - k'\theta^{n'-n}| < \theta^{-M|I_r|} \tag{3.20}$$

and hence, since either $k = k'\theta^{n'-n}$ or $|k - k'\theta^{n'-n}| > b^{-|I_r|}$,

$$b^{n'-n} | k' \tag{3.21}$$

(choosing M sufficiently large).

By hypothesis, for $k' \in \operatorname{supp}\hat{f}$, (3.21) may only hold for $n' - n < N/K$. Hence, as one easily verifies,

$$(3.19) \le C|I_r| \frac{N}{K} \|f\|_2^2. \tag{3.22}$$

Substitution of (3.22) in (3.17) yields

$$\left\| \sum_1^N f(\theta^n x) \right\|_{L^2(0,1)} \le CNK^{-1/2}\|f\|_2 + C \sum_{r=M}^{R} |I_r| \cdot K^{-1/4} \|f\|_2 \tag{3.23}$$
$$= CK^{-1/4}N\|f\|_2$$

and hence (3.14).

This concludes the proof of Theorem 1.3 for rational dilation θ. The argument for general algebraic θ is more involved (see [B2]) but the general mechanism is similar, a combination of applications of the maximal ergodic theorem and almost orthogonality considerations.

4. Riemann sums

Theorem 1.10 reduces to proving the following L^2-maximal inequality,

$$\left\| \sup_{N=2^{2^s}} \frac{1}{N} \left| \sum_1^N R_n f \right| \right\|_2 \le C\|f\|_2 \tag{4.1}$$

for $f \in L^2(\mathbf{T})$. With the notations from the introduction (1.9)

$$\frac{1}{N} \sum_1^N R_n f(x) = \sum_{k\in \mathbf{Z}} \frac{d(k;N)}{N} \hat{f}(k) e^{2\pi i k x}, \tag{4.2}$$

where $d(k;N)$ stands for the number of divisors of k less than N. Let for $z \in \mathbf{Z}_+$ the function χ_z be defined as

$$\chi_z(n) = \begin{cases} 1 \text{ for } z \mid n, \\ 0 \text{ otherwise.} \end{cases} \tag{4.3}$$

Thus χ_z is the indicator function of $z.\mathbf{Z}$.

Denote

$$\mathcal{P} = \{\text{prime numbers}\},$$

$$\mathcal{P}^* = \bigcup_{j=1}^{\infty} \mathcal{P}^j. \tag{4.4}$$

Clearly

$$d(k;N) = \int_1^N \Big\{ \prod_{\substack{z \in \mathcal{P}^* \\ z \nmid k}} (1 - \chi_z) \Big\} dn, \tag{4.5}$$

where integration on $\mathbf{Z}$ refers to the discrete counting measure. The product in (4.5) may be restricted to $z \leq N$. The problem of evaluating (4.5) comes from the fact that the $(\chi_z)_{z \in \mathcal{P}^*, z \leq N}$ are only "independent" in a limited sense when restricted to the interval $[1, N]$. This is the source of difficulties.

Consider the multipliers

$$\mu_k^{(N)} = \prod_{\substack{z \in \mathcal{P}^*, z \leq N \\ z \nmid k}} \left(1 - \frac{1}{z}\right) = \prod_{z \in \mathcal{P}^*, z \leq N} \left(\left(1 - \frac{1}{z}\right) + \frac{1}{z} \chi_z(k)\right). \tag{4.6}$$

Clearly

$$\left(1 - \frac{1}{z}\right) + \frac{1}{z} \chi_z(k) = |\hat{\mu}_z(k)|^2 \tag{4.7}$$

where μ_z is the probability measure on $\mathbf{T}$ defined by

$$\mu_z = \left(1 - \frac{1}{z}\right)^{1/2} \delta_0 + \frac{1}{z}\left(1 - \left(1 - \frac{1}{z}\right)^{1/2}\right)(\delta_0 + \delta_{1/z} + \delta_{2/z} + \ldots + \delta_{(z-1)/z}). \tag{4.8}$$

Again, δ_x denotes the Dirac measure at the point x.

It follows from (2.7), thus Rota's theorem, that

$$\left\| \sup_N \left| \sum_{k \in \mathbf{Z}} \mu_k^{(N)} \hat{f}(k) e^{2\pi i k x} \right| \right\|_2 \leq C \|f\|_2. \tag{4.8}$$

Our aim is to prove that

$$\sum_{N=2^{2^s},1\le s\le\infty}\left|\frac{d(k;N)}{N}-\mu_k^{(N)}\right|^2<C \tag{4.9}$$

uniformly for all frequencies k. From the discussion in section 2, this will conclude the proof of (4.1).

In what follows, we will be mainly concerned with evaluation of measures of intersections of "almost independent" sets.

LEMMA 4.10. *Let $\{a_j\}$ be a sequence of numbers in the interval* $[0,1]$. *Then, for $k=1,2,\dots$, there is the following inequality*:

$$\left|\prod(1-a_j)-\sum_{|S|<k}(-1)^{|S|}\prod_{j\in S}a_j\right|\le\sum_{|S|=k}\prod_{j\in S}a_j. \tag{4.11}$$

PROOF. By induction on the number of a_j's.

LEMMA 4.12. $$\frac{d(k;N)}{N}\le c\mu_k^{(N)}.$$

PROOF. Define

$$Q_s=[2^{2^{s-1}},2^{2^s}]\cap\{z\in\mathcal{P}^*\,|\,z\nmid k\}. \tag{4.13}$$

Thus

$$\sum_{z\in Q_s}\frac{1}{z}\le\sum_{\substack{p\in\mathcal{P}\\ 2^{s-1}\le\log p\le 2^s}}\frac{1}{p}+C\le C. \tag{4.14}$$

Fix s_*, to be specified, and estimate

$$\begin{aligned}&\frac{1}{N}\int_1^N\prod_{s=1}^{s_*}\prod_{z\in Q_s}(1-\chi_z)\\ &\le\frac{1}{N}\int_1^N\prod_{s=1}^{s_*}\left\{\sum_{\substack{A\subset Q_s\\|A|<k_s}}(-1)^{|A|}\prod_{z\in A}\chi_z+\sum_{\substack{A\subset Q_s\\|A|=k_s}}\prod_{z\in A}\chi_z\right\}dn\end{aligned} \tag{4.15}$$

using (4.10). Here $\{k_s\,|\,s\le s_*\}$ has to be specified also.

Expanding the integrand, one gets a sum of at most

$$\prod_{s=1}^{s_*}\binom{|Q_s|}{k_s}\le 2^{\sum_{s=1}^{s_*}k_s2^s} \tag{4.16}$$

terms, each of which is of the form $\pm\chi_z$, for some $z \in \mathbf{Z}_+$ (obviously $\chi_{z_1}\chi_{z_2} = \chi_{z_1 z_2}$). Clearly, one always has

$$\left| \frac{1}{N} \int_1^N \chi_z - \frac{1}{z} \right| \le \frac{1}{N}. \tag{4.17}$$

Hence, by (4.16), (4.15) equals

$$\prod_{s=1}^{s_*} \left\{ \sum_{\substack{A \subset Q_s \\ |A| < k_s}} (-1)^{|A|} \prod_{z \in A} \frac{1}{z} + \sum_{\substack{A \subset Q_s \\ |A| = k_s}} \prod_{z \in A} \frac{1}{z} \right\} \tag{4.18}$$

up to an approximation of

$$\frac{1}{N} 2^{\sum_{s=1}^{s_*} k_s 2^s}. \tag{4.19}$$

Again by (4.10) and by (4.14)

$$(4.18) \le \prod_{s=1}^{s_*} \left\{ \prod_{z \in Q_s} \left(1 - \frac{1}{z}\right) + 2 \sum_{\substack{A \subset Q_s \\ |A| = k_s}} \prod_{z \in A} \frac{1}{z} \right\}, \tag{4.20}$$

$$\sum_{\substack{A \subset Q_s \\ |A| = k_s}} \prod_{z \in A} \frac{1}{z} \le \frac{1}{k_s!} \left(\sum_{z \in Q_s} \frac{1}{z} \right)^{k_s} < \left(\frac{c}{k_s} \right)^{k_s}, \tag{4.21}$$

hence

$$(4.18) \le \prod_{s=1}^{s_*} \left(1 + \left(\frac{c}{k_s}\right)^{k_s}\right) \prod_{s=1}^{s_*} \prod_{z \in Q_s} \left(1 - \frac{1}{z}\right). \tag{4.22}$$

Put

$$k_s = s_* - s + 1 \qquad \text{for } 1 \le s \le s_*. \tag{4.23}$$

If $2^{2^{\bar{s}-1}} < N \le 2^{2^{\bar{s}}}$, we let $s_* = \bar{s} - 10$. Thus

$$\sum_{s=1}^{s_*} k_s 2^s \le 2^{s_*} \sum_1^\infty \frac{t+1}{2^t} < 5 \cdot 2^{s_*} < \frac{1}{2} 2^{\bar{s}-1} \tag{4.24}$$

and

$$(4.19) < N^{-1/2}. \tag{4.25}$$

It follows from (4.15), (4.22) that

$$\frac{d(k;N)}{N} \le \prod_1^\infty \left(1 + \left(\frac{C}{s}\right)^s\right) \prod_{s=1}^{s_*} \prod_{z\in Q_s} \left(1 - \frac{1}{z}\right) + N^{-1/2} \tag{4.26}$$

$$\le C \cdot \mu_k^{(N)} + N^{-1/2}$$

$$\le C \cdot \mu_k^{(N)}, \tag{4.27}$$

since clearly

$$\mu_k^{(N)} \ge C \prod_{p\in\mathcal{P}, p\le N} \left(1 - \frac{1}{p}\right) > \frac{C}{\log N}. \tag{4.28}$$

This proves the lemma. ■

We have to estimate the differences

$$\frac{d(k;N)}{N} - \mu_k^{(N)}$$

appearing in (4.9). Write, using the notations of the previous lemma and (4.10),

$$\left| \prod_{s=1}^{s_*} \prod_{z\in Q_s} (1 - \chi_z) - \prod_{s=1}^{s_*-1} \prod_{z\in Q_s} (1 - \chi_z) \left\{ \sum_{A\subset Q_{s*}, |A| \le k_{s*}} (-1)^{|A|} \prod_{z\in A} \chi_z \right\} \right|$$

$$\le \sum_{A\subset Q_{s*}, |A| = k_{s*}} \prod_{z\in A} \chi_z \tag{4.29}$$

and iterating

$$\left| \prod_{s=1}^{s_*} \prod_{z\in Q_s} (1 - \chi_z) - \prod_{s=1}^{s_*} \left\{ \sum_{A\subset Q_s, |A| \le k_s} (-1)^{|A|} \prod_{z\in A} \chi_z \right\} \right|$$

$$\le \sum_{A\subset Q_{s*}, |A| = k_{s*}} \prod_{z\in A} \chi_z + \left(\sum_{A\subset Q_{s*-1}, |A| = k_{s*-1}} \prod_{z\in A} \chi_z \right) \left(1 + \sum_{A\subset Q_{s*}, |A| = k_{s*}} \prod_{z\in A} \chi_z \right)$$

$$+ \left(\sum_{A\subset Q_{s*-2}, |A| = k_{s*-2}} \prod_{z\in A} \chi_z \right) \left(1 + \sum_{A\subset Q_{s*-1}, |A| = k_{s*-1}} \prod_{z\in A} \chi_z \right)$$

$$\times \left(1 + \sum_{A\subset Q_{s*}, |A| = k_{s*}} \prod_{z\in A} \chi_z \right) + \cdots$$

(4.30)

Integrate $(1/N)\int_1^N$ and observe that

$$\frac{1}{N}\int_1^N \chi_z dn \leq \frac{1}{z}$$

when evaluating the contribution of the right side of (4.30).

Using (4.19), (4.25) and the same approximation argument as in Lemma 4.12, it follows that

$$\left|\frac{1}{N}\int_1^N \prod_{s=1}^{s_*}\prod_{z\in Q_s}(1-\chi_z)dn - \prod_{s=1}^{s_*}\left\{\sum_{A\subset Q_s,|A|\leq k_s}(-1)^{|A|}\prod_{z\in A}\frac{1}{z}\right\}\right|$$

$$\leq N^{-1/2} + \sum_{t\leq s_*}\left(\sum_{A\subset Q_t,|A|=k_t}\prod_{z\in A}\frac{1}{z}\right)\left(1+\sum_{A\subset Q_{t+1},|A|=k_{t+1}}\prod_{z\in A}\frac{1}{z}\right)\cdots \tag{4.31}$$

$$\times\left(1+\sum_{A\subset Q_{s*},|A|=k_{s*}}\prod_{z\in A}\frac{1}{z}\right).$$

Hence, again using Lemma 4.10 and (4.23)

$$\left|\frac{1}{N}\int_1^N \prod_{s=1}^{s_*}\prod_{z\in Q_s}(1-\chi_z)dn - \prod_{s=1}^{s_*}\prod_{z\in Q_s}\left(1-\frac{1}{z}\right)\right|$$

$$\leq N^{-1/2} + 2\sum_{t\leq s_*}\left[1+\left(\frac{C}{k_{t+1}}\right)^{k_{t+1}}\right]\left[1+\left(\frac{C}{k_{t+2}}\right)^{k_{t+2}}\right]\cdots \tag{4.32}$$

$$\times\left[1+\left(\frac{C}{k_{s_*}}\right)^{k_{s_*}}\right]\frac{1}{k_t!}\left(\sum_{z\in Q_t}\frac{1}{z}\right)^{k_t}$$

$$\leq C\sum_{t\leq s_*}4^{-(s_*-t)}\left(\sum_{z\in Q_t}\frac{1}{z}\right) + N^{-1/2}.$$

Since

$$\left|\frac{1}{N}\int_1^N\left\{\prod_{\substack{x\in\mathcal{P}^*\\ z\nmid k}}(1-\chi_z)\right\}dn - \frac{1}{N}\int_1^N\left\{\prod_{s=1}^{s_*}\prod_{z\in Q_s}(1-\chi_z)\right\}dn\right| \leq \sum_{z\in Q_{s*+1}\cup\cdots\cup Q_{\bar{s}}}\frac{1}{z} \tag{4.33}$$

and similarly

$$\left|\mu_k^{(N)} - \prod_{s=1}^{s_*}\prod_{z\in Q_s}\left(1-\frac{1}{z}\right)\right| \leq \sum_{z\in Q_{s*+1}\cup\cdots\cup Q_{\bar{s}}}\frac{1}{z} \tag{4.34}$$

it follows from (4.32) and the fact that $\bar{s} - s_* = 10$

$$\left| \frac{d(k;N)}{N} - \mu_k^{(N)} \right| \le C \sum_{t \le \bar{s}} 4^{-(\bar{s}-t)} \left(\sum_{z \in Q_t} \frac{1}{z} \right) + N^{-1/2}. \tag{4.35}$$

Here, the integer $\bar{s}$ is defined by

$$N = 2^{2^{\bar{s}}}. \tag{4.36}$$

Also, by (4.12)

$$\left| \frac{d(k;N)}{N} - \mu_k^{(N)} \right| \le C \sum_{t \le \bar{s}} 2^{-(\bar{s}-t)} \prod_{\substack{z \in Q_{t'} \\ t' < t}} \left(1 - \frac{1}{z}\right)^{1/2} \left(\sum_{z \in Q_t} \frac{1}{z} \right)^{1/2} + N^{-1/2} \tag{4.37}$$

using the inequality

$$\min\left(\sum_t \alpha_t, \beta \right) \le \sum_t (\alpha_t . \beta)^{1/2} \tag{4.38}$$

for $\alpha_t \ge 0$, $\beta \ge 0$.

Hence, the left side of (4.9) is bounded by

$$C \sum_{s=1}^{\infty} \sum_{t \le s} 2^{-(s-t)} \prod_{\substack{z \in Q_{t'} \\ t' < t}} \left(1 - \frac{1}{z}\right) \cdot \left(\sum_{z \in Q_t} \frac{1}{z} \right) \tag{4.39}$$

$$\le C \sum_{t=1}^{\infty} \prod_{\substack{z \in Q_{t'} \\ t' < t}} \left(1 - \frac{1}{z}\right) \cdot \left(\sum_{z \in Q_t} \frac{1}{z} \right). \tag{4.40}$$

By (4.14), this last expression is bounded by an absolute constant.

The proof of (4.9) and hence Theorem 1.10 is thus completed.

5. An L^2-maximal inequality

The following inequality is specifically L^2 and will be used in the proof of Theorem 1.16.

Proposition 5.1. *Let $\{\mu_j\}_{j \ge 0}$ be a sequence of probability measures on* **R** *and consider for each $n \in \mathbf{Z}$ a sequence $(\beta_{n,j})_{j \ge 0}$ of scalars satisfying*

$$\sum_{j=0}^{\infty} |\beta_{n,j}|^2 \le 1. \tag{5.2}$$

Then, for $f \in L^2(\mathbf{T})$, the following inequality holds

$$\left\| \sup_j \left| \sum_{n\in\mathbf{Z}} \hat{f}(n) \left\{ \sum_{k\le j} \beta_{n,k} \mathrm{Im}\, \hat{\mu}_k(n) \prod_{l<k} \mathrm{Re}\, \hat{\mu}_l(n) \right\} e^{2\pi i n\theta} \right| \right\|_2 \le C\|f\|_2 \tag{5.3}$$

where C is an absolute constant.

Proof. Let μ_j be an image measure under independent random variables $\xi_j : \Omega \to \mathbf{R}$, on the probability space Ω. Thus

$$\int \cos n\xi_j(\omega) d\omega = \mathrm{Re}\, \hat{\mu}_j(n), \tag{5.4}$$

$$\int \sin n\xi_j(\omega) d\omega = \mathrm{Im}\, \hat{\mu}_j(n). \tag{5.5}$$

Denote $D = \{1, -1\}^{\mathbf{N}}$ with normalized product measure and generic element $\epsilon = (\epsilon_1, \epsilon_2, \dots)$. Define

$$U_j^{(n)}(\omega, \epsilon_{j+1}, \epsilon_{j+2}, \dots) = e^{2\pi i n \sum_{k>j} \epsilon_k \xi_k(\omega)}. \tag{5.6}$$

Put

$$F(\theta, \epsilon, \omega) = \sum_{n\in\mathbf{Z}} \left[\sum_{j\ge 0} \epsilon_j \beta_{n,j} U_j^{(n)} \right] \hat{f}(n) e^{2\pi i n\theta} \tag{5.7}$$

for which obviously

$$\|F\|^2_{L^2(d\theta\otimes d\epsilon)} = \sum_{n\in\mathbf{Z}} \left(\int \left| \sum_{j\ge 0} \epsilon_j \beta_{n,j} U_j^{(n)} \right|^2 d\epsilon \right) |\hat{f}(n)|^2$$

$$= \sum_{n\in\mathbf{Z}} \left(\sum_{j\ge 0} |\beta_{n,j}|^2 \right) |\hat{f}(n)|^2 \le \|f\|_2^2.$$

Denote $\mathbf{E}^{(j)}$ the expectation operator on D with respect to the variables $\epsilon_j, \epsilon_{j+1}, \dots$. Hence, for fixed $\omega \in \Omega$, we have

$$\mathbf{E}^{(j)} \left[\sum_{k\ge 0} \epsilon_k \beta_{n,k} U_k^{(n)} \right] = \sum_{k\ge j} \epsilon_k \beta_{nk} U_k^{(n)} \tag{5.8}$$

and, applying the martingale maximal inequality (2.6) in the ϵ-variable, it clearly follows that

$$(5.9)\qquad \left\| \sup_j \left| \sum_{n\in\mathbf{Z}} \left[\sum_{k<j} \epsilon_k \beta_{n,k} U_k^{(n)} \right] \hat{f}(n) e^{2\pi i n\theta} \right| \right\|_{L^2(d\theta\otimes ds\otimes d\omega)} \le C\|f\|_2.$$

Fixing ϵ,ω, shift θ to $\theta - \sum_{j\ge 0} \epsilon_j \xi_j(\omega)$. After this change of variable, (5.9) becomes, because of (5.6),

$$(5.10)\qquad \left\| \sup_j \left| \sum_{n\in\mathbf{Z}} \left[\sum_{k<j} \epsilon_k \beta_{nk} e^{-2\pi i n \sum_{l\le k} \epsilon_l \xi_l(\omega)} \right] \hat{f}(n) e^{2\pi i n\theta} \right| \right\|_{L^2(d\theta d\epsilon d\omega)} \le C\|f\|_2.$$

Bringing the $\epsilon - \omega$ integration inside the supremum, one obtains

$$(5.11)\qquad \iint \epsilon_k \epsilon^{-2\pi i n \sum_{l\le k} \epsilon_l \xi_l(\omega)} d\epsilon\, d\omega = \int \epsilon_k \prod_{l\le k} \hat{\mu}_l(\epsilon_l n) d\epsilon = i \operatorname{Im} \hat{\mu}_k(n) \prod_{l<k} \operatorname{Re} \hat{\mu}_l(n)$$

and hence (5.3).

COROLLARY 5.12. *Let $\{\theta_j\}_{j\ge 0}$ be a sequence in* $\mathbf{T}$ *and $(\beta_{n,j})_{j\ge 0}$ scalar sequences fulfilling* (5.2). *Then*

$$(5.13)\qquad \left\| \sup_j \left| \sum_{n\in\mathbf{Z}} \hat{f}(n) \left\{ \sum_{k\le j} \beta_{n,k} \left(\sin n\theta_k \prod_{l<k} \cos n\theta_l \right) \right\} e^{2\pi i n\theta} \right| \right\|_2 \le C\|f\|_2.$$

PROOF. Apply (5.1) with $\mu_j = \delta_{\theta_j/2\pi}$.

REMARK. If we choose in particular

$$(5.14)\qquad \beta_{n,k} = \sin n\theta_k \prod_{l<k} \cos n\theta_l$$

inequality (5.13) holds on L^p, $p>1$, as a consequence of (2.7). In this sense, it may be seen as an L^2-generalization.

COROLLARY 5.15. *Let $\{\theta_j\}_{j\ge 0}$ be a sequence in* $\mathbf{T}$ *and φ a differentiable function on* $[0,1]$ *satisfying*

$$(5.16)\qquad |\varphi'(t)| < Ct^{\epsilon}$$

for some $\epsilon > 0$. Then one has the inequality

$$(5.17)\qquad \left\| \sup_j \left| \sum_{n\in\mathbf{Z}} \hat{f}(n) \varphi\left(\prod_{k\le j} |\cos n\theta_k| \right) e^{2\pi i n\theta} \right| \right\|_2 \le C\|f\|_2.$$

PROOF. It follows from the middle-value theorem and (5.16) that for $0\le x\le 1$

$$(5.18)\qquad |(\varphi(x) - \varphi(x\cdot|\cos n\theta_k|)| \le C|x|^{1+\epsilon}(1 - |\cos n\theta_k|) \le C|x|^{1+\epsilon} \sin^2 n\theta_k.$$

Write $\varphi\left(\prod_{k\le j}|\cos n\theta_k|\right)$ as a difference sum

$$\varphi(|\cos n\theta_1|) + \sum_{k\le j}\left[\varphi\left(\prod_{l\le k}|\cos n\theta_l|\right) - \varphi\left(\prod_{l<k}|\cos n\theta_l|\right)\right] \tag{5.19}$$

where, by (5.18), the kth increment is bounded by

$$C\prod_{l<k}|\cos n\theta_l|^{l+\epsilon}\cdot\sin^2 n\theta_k. \tag{5.20}$$

We are then led to a statement of the form (5.13), for scalars $\beta_{n,k}$ satisfying

$$|\beta_{n,k}| < C\cdot\prod_{l<k}|\cos n\theta_l|^{\epsilon}\cdot|\sin n\theta_k|. \tag{5.21}$$

These sequences are for individual n square summable with a uniform bound (only dependent on ϵ).

Further applications of Proposition 5.1 will be given in section 7.

6. Proof of Theorem 1.16

To prove part (a) if ξ not algebraic, use Proposition 2.16(b), where

$$\lambda_n^{(j)} = \prod_{k=1}^{j}\cos 2\pi\xi^k n. \tag{6.1}$$

Thus we show that for some $\delta > 0$, $\delta = 1$ say, the entropy numbers $N_f(\delta)$ are not uniformly bounded for f ranging in the unit ball of $L^2(\mathbf{T})$. Fix an integer $s \ge 1$ and consider functions f of the form

$$f = 2^{-s/2}\sum_{t=1}^{2^s} e^{2\pi i n_t\theta}, \tag{6.2}$$

where the n_t are well chosen frequencies. Observe that because of the hypothesis on ξ, the map

$$\mathbf{Z}\to\mathbf{T}^s : N\mapsto \{\xi^k n \mid 1\le k\le s\} \tag{6.3}$$

has dense image. Hence, also the map

$$\mathbf{Z}\to[-1,1]^s : n\mapsto \{\cos 2\pi\xi^k n \mid 1\le k\le s\} \tag{6.4}$$

has dense image.

Let the map $\{1,2,\dots,2^s\}\to\{1,-1\}^s : t\to\epsilon^t = (\epsilon_1^t,\dots,\epsilon_s^t)$ be one to one. From what precedes, one may, for each $1\le t\le 2^s$, introduce an integer n_t satisfying

$$\begin{aligned}&\cos 2\pi\xi n_t \approx \epsilon_1^t,\\ &\cos 2\pi\xi^k n_t = \epsilon_{k-1}^t\epsilon_k^t \qquad \text{for } 2 \le k \le s,\end{aligned} \tag{6.5}$$

where $\approx$ stands for an arbitrary good approximation. Thus, by (6.5),

$$\prod_{k=1}^{j} \cos 2\pi\xi^k n_t \approx \epsilon_j^t \qquad (1 \le j \le s) \tag{6.6}$$

and hence for $j \neq j'$

$$\begin{aligned}&\left\| 2^{-s/2}\sum_1^{2^s}\lambda_t^{(j)}e^{2\pi i n_t\theta} - 2^{-s/2}\sum_1^{2^s}\lambda_t^{(j')}e^{2\pi i n_t\theta}\right\|_2\\ &= s^{-s/2}\left(\sum_1^{2^s}\left|\prod_{k=1}^{j}\cos 2\pi\xi^k n_t - \prod_{k=1}^{j'}\cos 2\pi\xi^k n_t\right|^2\right)^{1/2}\\ &\approx 2^{-s/2}\left(\sum_1^{2^s}|\epsilon_j^t - \epsilon_{j'}^t|^2\right)^{1/2}\\ &= \left(\int_{\{1,-1\}^s}|\epsilon_j - \epsilon_{j'}|^2 d\epsilon\right)^{1/2} = \sqrt{2}.\end{aligned} \tag{6.7}$$

Hence, for f given by (6.2)

$$N_f(1) \ge s \tag{6.8}$$

which proves 1.16(a).

Part (b) will be mainly a consequence of (5.13).

Let ξ satisfy the equation

$$a_{d-1}\xi^{d-1} + a_{d-1}\xi^{d-2} + \cdots + a_1\xi + a_0 = 0 \tag{6.9}$$

where $a_0, a_1, \ldots, a_{d-1} \in \mathbf{Z}$, $(a_1, a_1, \ldots, a_{d-1}) = 1$.

The fact that ξ is algebraic is used in the following.

LEMMA 6.10. *There is a positive integer r such that for all $k \ge d$, the inequality*

$$1 - \prod_{j=k+1}^{k+r}\cos 2\pi\xi^j n \le C\sum_{j=k-d}^{k+r+d}\sin^2 2\pi\xi^j n \tag{6.11}$$

holds, for all $n \in \mathbf{Z}$.

Its proof depends on the following simple fact.

LEMMA 6.12. *Let $d \geq 1$ be an integer and F a proper subspace of the vector space $\mathbf{Z}_2^d$ over $\mathbf{Z}^2$. There is an integer r such that if $(y_s)_{s \geq 1}$ is a sequence in $\mathbf{Z}_2$ such that*

$$(6.13)\qquad \begin{aligned} &(y_1, \dots, y_d) \in F, \\ &(y_2, \dots, y_{d+1}) \in F, \\ &(y_3, \dots, y_{d+3}) \in F, \\ &\quad\vdots \end{aligned}$$

then $y_{d+1} + \cdots + y_{d+r} = 0$.

PROOF OF LEMMA 6.12. There exists $d_0 < d$ and for $s > d_0$ coefficients $\alpha_{s,1}, \dots, \alpha_{s,d_0} \in \mathbf{Z}_2$ such that

$$(6.14)\qquad y_s = \alpha_{s,1} y_1 + \alpha_{s,2} y_2 + \cdots + \alpha_{s,d_0} y_{d_0}$$

whenever $(y_s)_{s \geq 1}$ fulfills (6.13). The coefficients $\alpha_{s,1}, \dots, \alpha_{s,d_0}$ only depend on F. To each d_0-tuple I of consecutive elements of $\mathbf{Z}_+$, associate the matrix

$$(6.15)\qquad (\alpha_{s,1}, \dots, \alpha_{s,d_0})_{s \in I} \in \mathbf{Z}_2^{d_0 \times d_0}.$$

Because these range in a finite set, there is I and an integer $q > 0$ such that $\min I > d$ and

$$(\alpha_{s,1}, \dots, \alpha_{s,d_0})_{s \in I} = (\alpha_{s,1}, \dots, \alpha_{s,d_0})_{s \in I+q}.$$

Hence, by (6.14)

$$(6.17)\qquad y_s = y_{s+q} \qquad \text{for } s \in I.$$

Shifting back, one sees that if $(y_s)_{s \geq 1}$ satisfies (6.13), then

$$(6.18)\qquad (y_d, y_{d+1}, \dots, y_{d+d_0-1}) = (y_{d+q}, y_{d+q+1}, \dots, y_{d+q+d_0-1})$$

and therefore $(y_s)_{s \geq d}$ is q-periodic. Take $r = 2q$.

PROOF OF LEMMA 6.10. If the left member of (6.11) is small, one gets integers u_j $(k - d < j < k + r + d)$ satisfying

$$(6.19)\qquad 2\xi^j n \approx u_j.$$

Since, by (6.9),

$$(6.20)\qquad a_0 \xi^j + a_1 \xi^{j+1} + \cdots + a_{d-1} \xi^{j+d-1} = 0$$

it follows from (6.19) that also

$$a_0 u_j + a_1 u_{j+1} + \cdots + a_{d-1} u_{j+d-1} = 0 \qquad (k - d < j \leq k + r). \tag{6.21}$$

Denote $y_j \in \mathbf{Z}_2$ the element u_j mod 2. Since not all $a_0, a_1, \ldots, a_{d-1}$ are multiples of 2, the relation $a_0 y_j + a_1 y_{j+1} + \cdots + a_{d-1} y_{j+d-1} = 0$ is non-trivial. Applying Lemma 6.12, it follows that $y_{k+1} + y_{k+2} + \cdots + y_{k+r} = 0$, where r is the integer given by (6.12). Equivalently, $u_{k+1} + u_{k+2} + \cdots + u_{k+r}$ is even and

$$\prod_{j=k+1}^{k+r} \cos \pi u_j = 1. \tag{6.22}$$

(6.19) implies in particular that $\prod_{j=k+1}^{k+r} \cos 2\pi\xi^j n$ is positive. Of course, one may then write

$$1 - \prod_{j=k+1}^{k+r} \cos 2\pi\xi^j n \leq \sum_{j=k+1}^{k+r} (1 - |\cos 2\pi\xi^j n|) \leq \sum_{j=k+1}^{k+r} \sin^2 2\pi\xi^j n. \tag{6.23}$$

This proves Lemma 6.10.

Lemma 6.24. *There is a constant C such that*

$$\|b\xi^{d-1}\| \leq C \max(\|b\|, \|b\xi\|, \ldots, \|b\xi^{d-2}\|, \|b\xi^d\|, \|b\xi^{d+1}\|, \ldots, \|b\xi^{2d-2}\|) \tag{6.25}$$

for any real number b. Here $\|\lambda\|$ stands for the fractional part of λ, i.e. $\operatorname{dist}(\lambda, \mathbf{Z})$.

Proof. By (6.9) and multiplying with powers of ξ

$$\begin{aligned} \|a_{d-1} b\xi^{d-1}\| &\leq C(\|b\| + \|b\xi\| + \cdots + \|b\xi^{d-2}\|) \\ \|a_{d-2} b\xi^{d-1}\| &\leq C(\|b\xi\| + \cdots + \|b\xi^{d-2}\| + \|b\xi^d\|) \\ &\vdots \\ \|a_0 b\xi^{d-1}\| &\leq C(\|b\xi^d\| + \cdots + \|b\xi^{2d-2}). \end{aligned} \tag{6.26}$$

Since $(a_0, a_1, \ldots, a_{d-1}) = 1$, it follows from Bezout's theorem that

$$c_0 a_0 + c_1 a_1 + \cdots + c_{d-1} a_{d-1} = 1 \tag{6.27}$$

for some integers $c_0, c_1, \ldots, c_{d-1}$.

By (6.16), (6.27)

$$\begin{aligned} \|b\xi^{d-1}\| &\leq C \max(\|a_0 b\xi^{d-1}\|, \ldots, \|a_{d-1} b\xi^{d-1}\|) \\ &\leq \text{right member of (6.25)}. \end{aligned} \tag{6.28}$$

We now come to the proof of (6.16) (b). Choose $r > d$ satisfying (6.11). We are considering the multipliers (6.1), i.e.

$$\lambda_n^{(j)} = \prod_{k=1}^{j} \cos 2\pi n \xi^k. \tag{6.29}$$

When proving the maximal inequality (1.17), it suffices of course to consider $j \in r\mathbf{Z}_+$, since the full maximal function is estimated by a sum of r of these restricted, after application of a multiplier $\prod_{k=1}^{\rho} \cos 2\pi n \xi^k$ $(0 \leq \rho < r)$.

The difference sequence equals

$$\lambda_n^{r(j+1)} - \lambda_n^{rj} = \prod_{k=1}^{jr} \cos 2\pi \xi^k n \left[1 - \prod_{k=rj+1}^{r(j+1)} \cos 2\pi \xi^k n \right] \tag{6.30}$$

$$= \sum_{l=-d}^{r+d} \left\{ \frac{1 - \prod_{k=rj+1}^{r(j+1)} \cos 2\pi \xi^k n}{\sum_{l=-d}^{r+d} \sin^2 2\pi \xi^{rj+l} n} \left(\prod_{k \leq rj, k \notin K_j^{(l)}} \cos 2\pi \xi^k n \right) \cdot \sin 2\pi \xi^{rj+l} n \right\} \times \prod_{k \in K_j^{(l)}} \cos 2\pi \xi^k n \cdot \sin 2\pi \xi^{rj+l} n, \tag{6.31}$$

where

$$K_j^{(l)} = \{ rs + l \mid s < j \text{ and } j - s \text{ even} \} \tag{6.32}$$

is clearly contained in $[1, rj]$ (because of the restriction $j - s$ even).

Fixing $l = -d, \ldots, r + d$, one has to estimate the maximal function corresponding to the multipliers

$$\sum_{j \leq j'} \beta_{n,j} \left(\prod_{k \in K_j^{(l)}} \cos 2\pi \xi^k n \right) \cdot \sin 2\pi \xi^{rj+l} n = \sum_{\substack{j \leq j' \\ j \text{ even}}} + \sum_{\substack{j \leq j' \\ j \text{ odd}}} \tag{6.33}$$

where $\beta_{n,j}$ is the factor between brackets in (6.31).

Inequality (5.13) will be applied separately to both sums on the right of (6.33). With the notations of (5.12), we have thus

$$\theta_s = 2\pi \xi^{2rs+l}, \qquad s = 1, 2, \ldots \tag{6.34}$$

for the even sum, which becomes then

$$\sum_{s < t} \beta_{n,2s} \sin n\theta_s \prod_{s' < s} \cos n\theta_{s'}. \tag{6.35}$$

Thus it only remains to check a uniform bound

$$\sum_s |\beta_{n,2s}|^2 < C. \tag{6.36}$$

By (6.11), (6.36) reduces to

$$\sum_s \sin^2 2\pi\xi^{2rs+l}n \cdot \prod_{\substack{k\le 2rs\\ k\notin K_{2s}^{(l)}}} \cos^2 2\pi\xi^k n < C \tag{6.37}$$

or, since always

$$|\cos x| \le e^{-(\sin^2 x)/2}, \tag{6.38}$$

$$C > \sum_s \sin^2 2\pi n\xi^{2rs+l}\cdot \exp\left\{-\sum_{k\le 2rs,\, k\notin K_{2s}^{(l)}} \sin^2 2\pi\xi^k n\right\}. \tag{6.39}$$

We use Lemma 6.24 to add the missing terms in the sum appearing in the exponent. Thus, by (6.25), letting $b = 2\xi^{k-d+1}n$,

$$\begin{aligned} &|\sin 2\pi\xi^k n| \le C\max(|\sin 2\pi\xi^{k-d+1}n|, \dots, |\sin 2\pi\xi^{k-1}n|,\\ &|\sin 2\pi\xi^{k+1}n|, \dots, |\sin 2\pi\xi^{k+d-1}n|). \end{aligned} \tag{6.40}$$

Since, by (6.32),

$$K_{2s}^{(l)} = \{2rs' + l \mid 1 \le s' < s\}, \tag{6.41}$$

the reader will easily check that

$$\sum_{k\le 2rs} \sin^2 2\pi\xi^k n \le C' \sum_{k\le 2rs,\, k\notin K_{2s}^{(l)}} \sin^2 2\pi\xi^k n. \tag{6.42}$$

Thus (6.39) is bounded by

$$\sum_{s\ge 1} \sin^2 2\pi n\xi^{2rs+l} \exp\left\{-c\sum_{s'<s} \sin^2 2\pi n\xi^{2rs'+l}\right\}, \tag{6.43}$$

which is an expression of the form

$$\sum_{s\ge 1} \delta_s \exp\left(-c\sum_{s'<s} \delta_{s'}\right) \tag{6.44}$$

where $0 \le \delta_s \le 1$. These are clearly uniformly bounded. This proves Theorem 1.16.

7. Further remarks

In this section, I give more applications of inequality (5.13) to sequences satisfying the pointwise ergodic theorem with respect to L^2-functions. These are sequences $\Lambda \subset \mathbf{Z}_+$ such that the averages

$$A_N f = \frac{1}{|\Lambda_N|} \sum_{n \in \Lambda_N} T^n f \tag{7.1}$$

converge almost everywhere, given an arbitrary dynamical system (Ω, B, μ, T), μ a probability measure, T measure preserving, and $f \in L^2(\mu)$. Here

$$\Lambda_N = \Lambda \cap [1, N] \tag{7.2}$$

which we assume nonvoid.

The sequences considered here are constructed in the following way. Fix a basis $d \in \mathbf{Z}$, $d > 2$ and expand $n \in \mathbf{Z}_+$ as

$$n = \sum_{j \geq 0} q_j \cdot d^j, \tag{7.3}$$

where $q_j = q_j(n) \in \{0, 1, \ldots, d-1\}$. Define for instance

$$\Lambda = \{n \in \mathbf{Z}_+ \mid q_j(n) \in \{0, 1\}, \text{ for all } j\}. \tag{7.4}$$

Proposition 7.5. *The sequence* Λ *defined by* (7.4) *satisfies the property considered above, at the beginning of this section.*

There are of course variants of Proposition 7.5. We do not intend to investigate them systematically here. The reader may wish to consult [B1] for more details on what follows.

To verify the convergence of the averages (7.1) in L^2, one uses the spectral theory and is reduced to check pointwise convergence of the associated sequence of trigonometric polynomials in $\mathbf{T}$,

$$p_n(\lambda) = \frac{1}{|\Lambda_N|} \sum_{n \in \Lambda_N} e^{2\pi i n \lambda}. \tag{7.6}$$

One easily verifies that if $|p_N(\lambda)|$ does not tend to zero for $N \to \infty$, then also

$$\prod_{j=0}^{n} \left| \frac{1 + e^{2\pi i d^j \lambda}}{2} \right| = \prod_{j=0}^{n} |\cos \pi d^j \lambda| \tag{7.7}$$

does not converge to zero for $n \to \infty$, hence

$$\sum_{j=0}^{\infty} \sin^2 \pi d^j \lambda < \infty. \tag{7.8}$$

(7.8) is equivalent to the condition

$$\lambda \in \bigcup d^{-j} \mathbf{Z}. \tag{7.9}$$

The polynomials $p_N(\lambda)$ are easily seen to converge for λ a d-adic rational.

The proof of (7.5) is obtained from a maximal variational inequality (see [B1] for more details), which has the following form:

$$\sum_{j=1}^{J} \left\| \max_{N_j \le N \le N_{j+1}} |A_N f - A_{N_j} f| \right\|_2 = o(J). \tag{7.10}$$

This is a uniform estimate for f taken of L^∞-norm ≤ 1 and $\{N_j\}$ any (sufficiently rapidly increasing) sequence of integers.

Of course, the first stage is to obtain a maximal function estimate

$$\left\| \sup_N |A_N f| \right\|_2 \le C \|f\|_2. \tag{7.11}$$

In fact, we will only prove (7.11). The proof of (7.10) is obtained by further elaboration of this argument and the only additional technicalities are due to the nature of the statement (7.10). From this point of view, the situation is analogous to the case of "arithmetical sets" studied in [B1]. Clearly, in proving (7.11), one may replace the segments Λ_N by sets

$$\Lambda_n' = \left\{ \sum_0^{n-1} q_j d^j \,|\, q_j = 0 \text{ or } q_j = 1 \right\}. \tag{7.12}$$

Thus, we are let to consider the multipliers

$$\frac{1}{2^n} \sum_{(q_j) \in \{0,1\}^n} e^{2\pi i (\sum q_j d^j)\lambda} = 2^{-n} \prod_0^{n-1} (1 + e^{2\pi i d^j \lambda}) \tag{7.13}$$

$$= \prod_0^{n-1} e^{i\pi d^j \lambda} \cdot \cos \pi d^j \lambda \tag{7.14}$$

$$= e^{i\pi (d^n - 1)/(d-1)} \prod_0^{n-1} \cos \pi d^j \lambda. \tag{7.15}$$

Following the method of [B1], the problem (7.11) is equivalent to statement (7.11) in the shift model $(\mathbf{Z}, S)$ (by transference) which is thus the maximal function inequality for the Fourier multipliers (7.15), i.e.

$$\left\| \sup_n \left| \int_0^1 \hat{f}(\lambda) \left[\prod_0^{n-1} e^{i\pi d^j \lambda} \cdot \cos \pi d^j \lambda \right] e^{2\pi i \lambda x} d\lambda \right| \right\|_2 \leq C \|f\|_2 \tag{7.16}$$

where $\| \ \|_2$ stands for the $l^2(\mathbf{Z})$-sequence norm.

Our main tool in proving (7.16) is the $(\mathbf{Z},\mathbf{T})$-version of (5.12) (which was a $(\mathbf{T},\mathbf{Z})$-result).

Corollary 7.17. *Let $\{\theta_j\}$ be a sequence in $\mathbf{R}$ and $(\beta_j(\lambda))_{j \geq 0}$ functions of $\lambda \in [0,1]$, satisfying a pointwise estimate*

$$\sum_{j=0}^{\infty} |\beta_j(\lambda)|^2 \leq 1. \tag{7.18}$$

Then, the following inequality holds for functions $f \in l^2(\mathbf{Z})$:

$$\left\| \sup_j \left| \sum_{k \leq j} \int_0^1 \hat{f}(\lambda) \left[\beta_k(\lambda) \sin \lambda\theta_k \prod_{l<k} \cos \lambda\theta_l \right] e^{2\pi i \lambda x} dx \right| \right\|_2 \leq C \|f\|_2. \tag{7.19}$$

This inequality may be proved in the same way as (5.12) and is in fact formally equivalent to it (by rescaling considerations). Consider the differences

$$2^{-n-1} \prod_0^n (1 + e^{2\pi i d^j \lambda}) - 2^{-n} \prod_0^{n-1} (1 + e^{2\pi i d^j \lambda})$$

$$= 2^{-n} \prod_0^{n-1} (1 + e^{2\pi i d^j \lambda}) \frac{e^{2\pi i d^n \lambda} - 1}{2} \tag{7.20}$$

$$= 2^{-n} \prod_0^{n-1} (1 + e^{2\pi i d^j \lambda}) \left(-\sin^2 \pi d^n \lambda + \frac{i}{2} \sin 2\pi d^n \lambda \right) \tag{7.21}$$

$$= e^{i\pi (d^n - 1)\lambda/(d-1)} \prod_0^{n-1} \cos \pi d^j \lambda \left(-\sin^2 \pi d^n \lambda + \frac{i}{2} \sin 2\pi d^n \lambda \right), \tag{7.22}$$

by (7.15).

The contribution of the terms

$$e^{i\pi (d^j - 1)\lambda/(d-1)} \prod_0^{n-1} \cos \pi d^j \lambda \cdot \sin^2 \pi d^n \lambda \tag{7.23}$$

can be taken care of immediately, using (7.19) and the same procedure as in section 6. Thus split the sum into even and odd terms and consider, for instance, the maximal function

$$\sum_{j \text{ even}} \left| \sum_{\substack{k \le j \\ k \text{ even}}} \int_0^1 \hat{f}(\lambda) \left[e^{i\pi(d^k-1)\lambda/(d-1)} \cdot \prod_0^{k-1} \cos \pi d^l \lambda \cdot \sin^2 \pi d^k \lambda \right] e^{2\pi i \lambda x} d\lambda \right|. \tag{7.24}$$

Put (k even)

$$\theta_k = \pi d^k, \tag{7.25}$$

$$\beta_k(\lambda) = e^{i\pi(d^k-1)\lambda/(d-1)} \cdot \prod_{\substack{l<k \\ l \text{ odd}}} \cos \pi d^l \lambda \cdot \sin \pi d^k \lambda. \tag{7.26}$$

To verify the bound

$$\sum_k |\beta_k(s)|^2 < C \tag{7.27}$$

use the fact that

$$\prod_{l<k, l \text{ odd}} |\cos \pi d^l \lambda| < \exp\left\{ - \sum_{l<k, l \text{ odd}} \sin^2 \pi d^l \lambda \right\} \tag{7.28}$$

$$< \exp\left\{ -c \sum_{1 \le l \le k} \sin^2 \pi d^l \lambda \right\}, \tag{7.29}$$

since

$$|\sin dx| < C_d |\sin x|. \tag{7.30}$$

Hence (7.19) implies (7.24).

Thus it remains to consider the contribution of the second term in (7.22), i.e.

$$\sup \left| \sum_{k \le j} \int_0^1 \hat{f}(\lambda) \left[2^{-k} \prod_{l=0}^{k-1} (1 + e^{2\pi i d^l \lambda}) \cdot \sin 2\pi d^k \lambda \right] e^{2\pi i x \lambda} d\lambda \right|. \tag{7.31}$$

At this point, we use the following trick. Write

$$\sum_{k \le j} 2^{-k} \prod_{l=0}^{k-1} (1 + e^{2\pi i d^l \lambda}) \cdot \sin 2\pi d^k \lambda \tag{7.32}$$

$$= \frac{1}{d-1} \left\{ \sum_{k<j} \prod_{l=0}^{k-1} \frac{1 + e^{2\pi i d^l \lambda}}{2} \left[d \sin 2\pi d^k \lambda - \frac{1 + e^{2\pi i d^k \lambda}}{2} \sin 2\pi d^{k+2} \lambda \right] \right\} \tag{7.33}$$

$$-\frac{1}{d-1}\sin 2\pi\lambda \tag{7.34}$$

$$+\frac{d}{d-1}\prod_{l=0}^{j-1}\frac{1+e^{2\pi i d^l\lambda}}{2}\cdot\sin 2\pi d^j\lambda. \tag{7.35}$$

To estimate the contribution to the maximal function of (7.35), for varying j, use a direct square function argument, i.e., putting

$$\begin{aligned}\beta_j(\lambda) &= \prod_0^{j-1}\frac{1+e^{2\pi i d^l\lambda}}{2}\cdot\sin 2\pi d^j\lambda\\ &= e^{i\pi(d^j-1)/(d-1)}\prod_0^{j-1}\cos\pi d^l\lambda\cdot\sin 2\pi d^j\lambda\end{aligned} \tag{7.36}$$

one has

$$\sum_j |\beta_j(\lambda)|^2 < C \tag{7.37}$$

and we write

$$\sup_j\left|\int\hat f(\lambda)\beta_j(\lambda)e^{2\pi i\lambda x}d\lambda\right| \le \left(\sum_j\left|\int\hat f(\lambda)\beta_j(\lambda)e^{2\pi i\lambda x}d\lambda\right|^2\right)^{1/2}$$

which $l^2(\mathbf{Z})$-norm may be estimated by Parseval's identity was

$$\left(\sum_j\int|\hat f(\lambda)|^2|\beta_j(\lambda)|^2 d\lambda\right)^{1/2} \tag{7.38}$$

$$\le C\left(\int|\hat f(\lambda)|^2 d\lambda\right)^{1/2} = C\|f\|_{l^2(\mathbf{Z})} \tag{7.39}$$

(this is the same argument as considered at the beginning of section 2). Next, consider (7.33). The kth summand can be expressed as

$$e^{i\pi(d^k-1)/(d-1)}\cdot\prod_0^{k-1}\cos\pi d^j\lambda\cdot\begin{cases} d\sin 2\pi d^k\lambda - \sin 2\pi d^{k+1}\lambda & (7.40)\\ +\dfrac{1-\cos 2\pi d^k\lambda}{2}\sin 2\pi d^{k+1}\lambda & (7.41)\\ -\dfrac{i}{2}\sin 2\pi d^k\lambda\cdot\sin 2\pi d^{k+1}\lambda & (7.42)\end{cases}$$

In the (7.42) terms, 2 sine factors appear. For the (7.41) terms, write $1 - \cos 2\pi d^k\lambda = 2\sin^2\pi d^k\lambda$, yielding 3 sine factors. The maximal function corresponding to the sums (7.41) and the sums (7.42) may therefore be estimated similarly as above, when dealing with (7.23),(7.24). Consider now (7.40) (making these differences appear was the point of the representation (7.32)). One has

$$|d \sin x - \sin dx| < C|\sin x|^3. \tag{7.43}$$

Hence, (7.40) is the product of a bounded factor and $\sin^3 2\pi d^k \lambda$. Thus also here, the same estimate (7.19) can be applied. This concludes the proof of (7.16). Observe that it fits again the general pattern discussed in section 2.

Also (7.10) is equivalent to its shift formulation. Adaptation of the previous argument to get a maximal variational inequality will be straightforward for a reader a bit familiar with Fourier analysis.

It should be mentioned that the sequences S introduced in this section are a special case of sequences generated by a certain iterative procedure in the sense of [Q]. Considering the polynomials $p_N(\lambda)$ given by (7.6) easily leads for such sequences to Riesz-product type expressions to which the methods of this paper may be applied.

References

[B1] J. Bourgain, *Pointwise ergodic theorems on arithmetic sets, with an appendix on return time sequences* (jointly with H. Furstenberg, Y. Katznelson, D. Ornstein), Publ. Math. IHES 69.

[B2] J. Bourgain, *The Riesz-Raikov theorem for algebraic numbers*, in Piatetski-Shapiro Festshrift, to appear (preprint IHES, 1989).

[B3] J. Bourgain, *Almost sure convergence and bounded entropy*, Isr. J. Math. **63** (1988), 79–97.

[B4] J. Bourgain, *Almost sure convergence in Ergodic theory*, in *Columbus Proc. on "Almost Everywhere Convergence,"* Academic Press, New York 1989, pp. 145–152.

[Je] B. Jessen, *On the approximation of Legesgue integrals by Riemann sums*, Ann. Math. **35** (1934), 248–251.

[Q] M. Queffélec, *Substitution Dynamical Systems—Spectral Analysis*, Springer-Verlag, Berlin, 1987.

[Ro] G. C. Rota, *An "Alternierende Verfahren" for general positive operations*, Bull. Am. Math. Soc. **58** (1962), 95–102.

[Ru] W. Rudin, *An arithmetic property of Reimann sums*, Proc. Am. Math. Soc. **15** (1964), 321–324.

[Sa] R. Salem, *Collected Works*, Hermann, Paris, 1967.

[Th] J. P. Thouvenot, *La convergence presque sûre des moyennes ergodiques suivant certaines sous-suites d'entiers*, Sem. Bourbaki, Vol. 1989-90. Exp. 719.